APOLOGY FOR THE PROOF OF THE RIEMANN HYPOTHESIS

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The Riemann hypothesis is the product of a renaissance in mathematics which occurred in the seventeenth century after more than a thousand years in which it lay dormant in libraries and monasteries. The spirit of the renaissance is captured in the Cartesian philosophy that the world's problems are solved by thought. There is another philosophy, which predominates in the present day, that the world's problem are solved by taking action. An appreciation of Cartesian philosophy is proposed as a means of making action more effective.

The Riemann hypothesis is a product of mathematical analysis. Analysis can be described in general terms as the application of thought as a preliminary to action. Effective thinking is not made in a vacuum. It requires hypotheses without which no valid conclusion can be drawn. Although there are many striking examples of analysis, the analysis which is made in mathematics surpasses all other forms of analysis in the extent and consistency of its logical structure. Other applications of analysis emulate the analysis which is made in mathematics when that analysis cannot be applied directly.

Since the goals of mathematical analysis are less immediate than other forms of analysis, it is instructive to search for them in the history of mathematics. Early Greek mathematicians were primarily geometers. The original goal of mathematical analysis was to discover the properties of the three–dimensional space in which all human activity takes place. Awareness that space has interesting properties is a necessary preliminary to the acceptance of such a goal. Evidence of such awareness is found in the architectural precision of Egyptian pyramids. The underlying goal of mathematical analysis as the study of three–dimensional space has not been lost in the present day. Awareness that space has interesting properties remains a key to the appreciation of mathematical analysis. The remarkable properties of space are conjectured to explain all physical and chemical phenomena which are observed within it. In a forseeable future the scientific experiments which are now performed in laboratories will be replaced by computer calculations.

The rebirth of mathematical analysis which occurred in the seventeenth century is exemplified by the lives of René Descartes (1596–1652), Pierre de Fermat (1601–1665), and Blaise Pascal (1623–1662). Cartesian analysis is characterized by the systematic use of numbers to describe objects in space. An appreciation of numbers, as they occur in Greek mathematics, is an inevitable consequence of Cartesian thought. The Euclidean algorithm is then seen as a fundamental contribution to analysis which was made in the fourth century BCE. This dating of the Euclidean algorithm does not contradict its earlier discovery in China since what is now meant is a pattern of thinking which Greek civilization has transmitted to the renaissance. For the first time there appeared the concept of an open society in which knowledge was not the privilege of an elite class but was made available in public lectures and was preserved in libraries. The Euclidean algorithm was applied under these favorable conditions to the properties of the positive integers. It was found that every positive integer admits an essentially unique factorization as a product of primes. The existence of an infinite number of primes was known.

The properties of positive integers are stimulating as a focus of interest for beginning students of mathematics. Since the positive integers are infinite in number, it is not possible to treat them by the methods applied to finite objects. The very existence of the positive integers is a hypothesis which cannot be verified from simpler hypotheses. It is an axiom on which the existence of mathematics is predicated.

Something similar occurs in the foundations of the Christian faith. In the fourth century the first Christian emperor, Konstantin, compelled the leaders of Christianity to settle their differences. They agreed on the Nicene Creed, which all Christians accept as a definition of their faith. This statement is convincing because of its clear logical structure. There are for a Christian three expressions of divine presence. The first is the divine creation of an orderly universe within which life is possible. The second is the divine message carried to mankind of what use is to be made of the creation. And the third is the expression of divine will in human beings as they treat each other. These truths are hypotheses on which conclusions are based, not conclusions from more fundamental hypotheses. A Christian cannot logically conclude that anyone who disagrees with his reading of these hypotheses is wrong. A Christian is however entitled to ask others whether they have better hypotheses to offer.

The comparison between mathematics and theology is instructive. Christian theology, like mathematics, is based on analysis. The axiomatic nature of Christian belief is presented with greater clarity than is the axiomatic nature of the positive integers. A major obstacle to overcome in teaching is the belief that numbers exist in some absolute sense. If this were true, then everything that one might want to know about them could be obtained by observation. But this is not possible. Not even the most powerful computer can be applied to a verification for all positive integers. A student of mathematics who does not appreciate the difficulty is unable to complete the simplest argument applying to all positive integers. He cannot for example show that

$$m + n = n + m$$

for all positive integers m and n. For the proof of such an assertion requires a prior decision about its meaning. Even the definition of addition cannot be made for all positive integers by the most sophisticated computer. In the language of theology the student is in a state of original sin which he cannot surmount without divine assistance which in this case is transmitted by a teacher.

The relationship between mathematics and theology, which now seems distant, was immediate in the seventeenth century. Even the simplest numbers, such as the positive integers, were seen to require an axiomatic treatment like the Christian faith. No logical argument can prove that Christian beliefs are true. But someone who accepts the Nicene Creed can logically base Christian beliefs on them. The existence of the positive integers cannot be proved. But once their existence is accepted as a hypothesis, a logical treatment of their properties is possible.

Mathematical analysis in the seventeenth century is motivated by the Cartesian reduction of space to numbers and by the axiomatic nature of the positive integers. The Euclidean algorithm assumes a central position when the history of mathematics is examined from that perspective. Another central position is assumed by the contributions to number theory made by Diophantus, a member of the Greek school in Alexandria in the third century. Not all of his books survived the destruction of the library of Alexandria in the seventh century. But those books which were saved by Muslim scholars indicate a knowledge of what was later seen to be a fundamental theorem of number theory. Namely every positive integer is the sum of four squares of integers. What does survive is the representation of some positive integers by a smaller number of squares. A prime for example is a sum of two squares if it is congruent to one modulo four. A prime is of the form

$$a^2 - ab + b^2$$

for integers a and b if it is congruent to one modulo six. Conditions for the representation of other positive integers are derived from the representation of primes. These investigations of the properties of positive integers were resumed in the seventeenth century. Fermat is the leading contributor to this research.

The stimulating effect of Christian theology on mathematics is illustrated by the life of Pascal. Christian theology is founded on the writings of Saint Augustin in the fourth century. The Jansenist movement of the seventeenth century captured what was seen as the essence of his teaching. Education was emphasized at the Jansenist monastery of Port–Royal near Paris. The teaching of Jansenist nuns was not restricted to the wealthy and noble, as was customary at that time, but was open to all who wanted to learn. The application of Jansenist principles to the teaching of mathematics in instructive. Pascal received a classical education in the humanities after the loss of his mother in childhood. When he was twelve, he learned by chance of the existence of a subject called geometry which is concerned with the properties of figures in space. The incomplete nature of the information he received stimulated him to reinvent the subject. Only then did his father supplied him with a copy of Euclid's *Elements*. Pascal became a major contributor to the mathematics of the seventeenth century. He continued to discover new theorems in geometry until his death. The education of Blaise Pascal is described with meticulous care by his older sister Jacqueline in the preface to Pascal's *Pensées*. A portrait of Jacqueline Pascal by the court painter, Philippe de Champaigne, hangs in the national museum on the site of the ancient abbey of Port–Royal. A visit to the museum is recommended for those who are overwhelmed by the difficulties in teaching mathematics. The secret of Jansenist success in teaching is a faith in the potentialities of human nature which permits the unfolding of talent when it exists.

The Cartesian reduction of space to numbers was effectively continued in England by Isaac Newton (1640–1727). The infinitesimal calculus as it applies to constructions in space was already known to Archimedes in the second century BCE. A major reformulation of the calculus results from a systematic use of numbers. For this purpose Newton had a supply of mathematical information which had been collected by monks in monasteries. The calculus is reformulated as a theory of functions. Formulae replace diagrams. The calculus is applied not only to constructions in space but also to the properties of numbers. The Newton polynomials

$$\frac{s(s-1)\dots(s+1-n)}{1\dots n}$$

exhibit a remarkable relationship between functions and positive integers. They lay the foundations for a theory of special functions which appears repeatedly in later applications of mathematics, and in particular to the Riemann hypothesis.

Newton is responsible for a dynamical interpretation of Cartesian coordinates. The coordinates are seen to describe the position of a point which moves around the origin of coordinates as a center. The application to planetary motion implements the conceptions of Johannes Kepler (1571–1630). For the description of motion Newton introduces momentum, which like position is determined by three Cartesian coordinates. Motion results when position changes in time under the influence of momentum, and momentum changes in time as determined by position. A mysterious force called gravity is invoked to act at a distance. These imaginative constructions have a convincing application to observed planetary motion.

The Newtonian interpretation of Cartesian philosophy was a source of inspiration for Francois Marie Arouet (1694–1778), better known as Voltaire. Newton was confident that the physical universe obeys natural laws which are subject to analysis. Voltaire searched for the natural laws of politics, as had Plato in the third century BCE. The conclusions obtained were surprising at a time when political power was justified by divine right. It was generally believed that effective government required a privileged class of those who were trained from birth for that purpose. Analysis indicated that greater stability in government would be obtained with the inclusion of all members of society. All that was needed was an effective mechanism for determining the will of the governed and causing changes in government. The successful application of analysis to politics needs to be kept in mind for appreciating progress in mathematical analysis.

The Newtonian interpretation of Cartesian philosophy is implemented in the mathematics of Leonard Euler (1707–1783), who learned of it through his association with the Bernoulli family in Basel. An early contribution to function theory is his discovery of the gamma function in 1730, which is a sequel to the theory of Newton polynomials. His great contribution to number theory is the discovery in 1737 of the classical zeta function. The resulting relationship between function theory and number theory is the underlying theme of the Riemann hypothesis. The first publication on the zeta function gives only the Euler product. The functional identity for the zeta function was not obtained until 1761. In the initial years of his career Euler was a member of the recently founded Russian academy of sciences in Saint Petersburg. He was then invited to the court of Frederick the Great in Berlin, where he remained until 1766. A proof that every positive integer is a sum of four squares was obtained in 1770 by Louis de Lagrange (1736–1813) after preparation by Euler.

The significance of the classical zeta function for the distribution of prime numbers must have been known to Euler and was stated by Lagrange, but the definitive formulation is due to Carl Friederich Gauss (1777–1855). The number of primes less than a given positive number x is approximated by the integral

$$\int_e^x \frac{dt}{\log(t)}.$$

Gauss made a contribution to the theory of the gamma function by a systematic treatment of the hypergeometric series introduced by Euler as a generalization of the Newton polynomials. The foundations of linear algebra are contained in his canonical form for square matrices which exhibits their invariant subspaces. He applied the resulting theory to the structure of the Fourier transformation on the integers modulo r for positive integers r. The results permitted Lejeune Dirichlet (1805–1859) to construct zeta functions which resemble the classical zeta function in Euler product and functional identity.

The evolution of political analysis created the setting for the evolution of mathematical analysis. A test of political science was posed in 1776 by the Declaration of Independence, which rejected English rule of the American colonies. The success of the experiment in government was still in doubt in 1830 when Alexis de Tocqueville gave his evaluation in *Democracy in America*. Jeffersonian democracy passed the test with the rejection of slavery following the American Civil War (1860–1865). The French Revolution (1789–1799) was unsuccessful by comparison. Political analysts, who might be compared to

Jefferson, such as Pierre Samuel du Pont de Nemours (1739–1813) were unable to prevent the execution of Louis XVI in 1793. When the great chemist Lavoisier was executed in 1794, his student Eleuthère Irénée du Pont de Nemours (1771–1834) emigrated to the United States. He founded a chemical company which supplied gunpowder to the Union Army in the Civil War and to the French and American Armies in the First World War. Both wars consolidate progress in democracy.

Fundamental contributions to mathematical analysis were made in the politically unstable times which followed the French Revolution. Joseph Fourier (1768–1830) created a function theory adapted to the needs of Fourier analysis in pure and applied mathematics. Denis Poisson (1781–1840) discovered an identity in Fourier analysis which eventually became known as the Poisson formula because of its usefulness in proving the functional identity of a zeta function. Augustin Cauchy (1789–1857) created a function theory which is adapted to the needs of zeta functions and which eventually became known as complex analysis. Carl Gustav Jacobi (1804–1851) determined the number of representations of a positive integer as a sum of four squares. Willian Rowan Hamilton (1805–1865) characterized the three–dimensional space of Descartes and Newton by embedding it in the skew–field of quaternions. The number of representations of a positive integer as a sum of four squares in equal to twenty–four times the sum of the odd divisors of n in the quaternion count of representations.

The applications of complex analysis to zeta functions are due to Bernhard Riemann (1826–1866). A zeta function is an analytic function $\zeta(s)$ of s in the complex plane with the possible exception of a singularity at one. If there is a singularity, the product

$$(s-1)\zeta(s)$$

is an analytic function of s in the complex plane when defined by continuity at one. The Euler product, which applies in the half–plane $\mathcal{R}s > 1$, implies the absence of zeros in the half–plane. The functional identity, which relates the function $\zeta(s)$ of s to the function $\zeta(1-s)$ of s, implies the absence of zeros in the half–plane $\mathcal{R}s < 0$ except for so called trivial zeros on the real axis caused by singularities of a gamma function factor in the functional identity. The values of the zeta function in the critical strip $0 < \mathcal{R}s < 1$ are obtained by analytic continuation, a procedure which does not determine zeros. The functional identity implies the symmetry of the zeros in the critical strip, or on its boundary, about the critical line $\mathcal{R}s = \frac{1}{2}$. The Riemann hypothesis is the conjecture, published in 1859, that all nontrivial zeros lie on the critical line. Riemann stated the conjecture for the classical zeta function and indicated the application to the Gauss estimate of the number of primes less than a given positive number.

The Riemann hypothesis is generally acknowledged by mathematicians to be the most important unsolved problem in mathematics. A comparison may be helpful in explaining the importance of the Riemann hypothesis to nonmathematicians. The Riemann hypothesis tests the merits of mathematical analysis in the same way as the Declaration of Independence tested the merits of political analysis. When the decision was made to create a government on democratic principles, it was not clear that the experiment would be successful because of the existence of slavery. When the American democracy was able at great cost to eliminate slavery, it demonstrated the political wisdom of its founders. The Riemann hypothesis is a similar test of the merits of mathematical analysis. The condition of mathematics before the proof of the Riemann hypothesis is comparable to the condition of the American democracy before the abolition of slavery. Mathematics without the Riemann hypothesis abounds in good intentions which are unfulfilled.

The proof of the Riemann hypothesis offers a perspective on the history of mathematics. Although the conjecture was conceived for its application to numbers, it has a deeper application to the properties of space. That space can be a line or a plane, but there is also the three–dimensional space of Descartes and Newton which Hamilton embedded in the four–dimensional space of quaternions. Three–dimensional space admits symmetries which are exhibited by a cube. Although a sphere can be turned into itself along any axis through its center, there are only twenty–four of these motions which turn an inscribed cube into itself.

Space also has less visible symmetries which are observed through their consequences. An analogy helps to understand the nature of such symmetries. A moving fan may seem to be standing still to a drowsy eye on a hot summer day. There are alternating patterns of two, three, and four blades. When the mind fails to perceive the details of motion, it decomposes the partial information received into elementary components by a process called Fourier analysis. Underlying symmetries of motion appear through the observation of incomplete data.

Symmetries of space appear through the observation of the motion of objects through space. This dynamic treatment of Cartesian space is the fundamental contribution of Isaac Newton. The origin of coordinates is like a sun around which planets are moving. But symmetries are not observed when moving objects are treated as points, as did Newton. Electrons moving around a nucleus are more difficult to observe than planets moving around the sun. The information received is so incomplete that only the most elementary Fourier components are identified. The symmetries of space are discovered indirectly as symmetries of functions defined in space. The difficulty in finding symmetry does not lie so much in the function concept as it does in the mechanism for describing symmetry in functions. This analytical geometry is the fundamental contribution of William Rowan Hamilton.

Analytical geometry is widely taught in conjunction with the infinitesimal calculus, but what is taught is inadequate for the purposes of describing symmetry. The more structured formulation of Hamilton by quaternions is considered too difficult for ordinary students. The additional effort needed is however rewarded by a discovery of the characteristic properties of Cartesian space. What is taught is a vector calculus in which addition and multiplication by scalars is supplemented by the dot and cross product of vectors. An additional fourth dimension is implicit since scalars are not vectors. Hamilton accepts scalars on an equal basis with vectors to form the four–dimensional space of quaternions. The fourth dimension is usually thought of as time for the purposes of calculation. The scalar component of quaternions is however only an adjunct to Cartesian space without other implications. It is not identical with the relativistic formulation of time which appears in electromagnetic theory. A relationship between the two concepts of time exists, but only at a higher level of algebra. Relativistic time resembles imaginary numbers when quaternions are seen as resembling real numbers.

A quaternion according to Hamilton is a linear combination

$$\xi = t + ix + jy + kz$$

of basic quaternions i, j, k, and 1 with real numbers as coefficients x, y, z, and t. The conjugate quaternion is

$$\xi^- = t - ix - jy - kz.$$

A quaternion ξ is the sum of a self–conjugate quaternion

$$t = \tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-$$

and a skew–conjugate quaternion

$$ix + jy + kz = \tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-.$$

Self–conjugate quaternions are added and multiplied as real numbers. Skew–conjugate quaternions are added as Cartesian vectors and are multiplied by self–conjugate quaternions treated as numbers. The product of two skew–conjugate quaternions is the sum of a self–conjugate quaternion, which is minus the dot product of two Cartesian vectors, and a skew–conjugate quaternion, which is the cross product of two Cartesian vectors. The multiplication of quaternions reformulates the vector analysis taught in courses on analytical geometry. The multiplication of quaternions satisfies the associative law

$$(\alpha\beta)\gamma = \alpha(\beta\gamma),$$

which holds for all quaternions α, β, and γ. The conjugation of quaternions satisfies the involutory law

$$(\xi\eta)^- = \eta^-\xi^-,$$

which holds for all quaternions ξ and η. The products

$$\xi^-\xi = x^2 + y^2 + z^2 + t^2 = \xi\xi^-$$

of a quaternion ξ with its conjugate are equal, and they are positive when the quaternion is nonzero. These properties of quaternions imply a characterization of Cartesian space since Euclidean spaces of larger dimension do not admit a multiplicative structure sharing these properties.

A unit quaternion is a quaternion ω such that

$$\omega^-\omega = 1.$$

The conjugate of the quaternion is then equal to its inverse. If ω is a unit quaternion, an automorphism of quaternions is defined by taking ξ into

$$\omega^-\xi\omega.$$

The automorphism commutes with the involution ξ into ξ^-. Self–conjugate quaternions are left fixed by the automorphism. Skew–conjugate quaternions map into skew–conjugate quaternions in such a way as to preserve the dot and cross products of vectors. The automorphism then acts as a rotation of Cartesian space. Every rotation of Cartesian space is determined by such an automorphism. But the unit quaternion defining a rotation is not unique since ω and $-\omega$ determine the same automorphism. A nontrivial rotation of Cartesian space rotates the space about an axis of vectors which are left fixed by the rotation. The vectors which lie on the axis lie in the direction of the skew–conjugate component of the unit quaternion ω or in the opposite direction. The self–conjugate component of ω is the cosine of one–half the angle of rotation. The unit quaternions which define a nontrivial rotation are then determined by the axis of rotation and the angle of rotation. The unit quaternions which determine the trivial rotation, which leaves every vector fixed, are the self–conjugate quaternions 1 and -1.

The rotations which map a cube into itself are easily computed. The cube is placed with its center at the origin of coordinates and its vertices at the eight vectors whose Cartesian coordinates are one and minus one. The centers of the edges of the cube are at the eight vectors which have one Cartesian coordinate equal to zero and the two other coordinates equal to one or minus one. The centers of the faces of the cube are the six vectors which have two Cartesian coordinates equal to zero and the other coordinate equal to one or minus one. A nontrivial rotation of the cube into itself leaves fixed a pair of opposite vertices, or a pair of opposite centers of edges, or a pair of opposite centers of faces. If the rotation leaves fixed a pair of opposite vertices, the cosine of one–half the angle of rotation is one–half or minus one–half since the rotation is one–third or two–thirds of a complete revolution. If the rotation leaves fixed a pair of opposite edges, the cosine of one–half the angle of rotation is zero since the rotation is one–half of a complete revolution. If the rotation leaves fixed a pair of opposite centers of faces, the cosine of one–half the angle of revolution is zero when the rotation is one–half of a complete revolution and is a positive or negative square root of one–half when the rotation is one–quarter or three–quarters of a complete revolution.

The eight nontrivial rotations of the cube which leave fixed a pair of opposite vertices are determined by the sixteen unit quaternions

$$\pm\tfrac{1}{2} \pm \tfrac{1}{2}i \pm \tfrac{1}{2}j \pm \tfrac{1}{2}k.$$

The six nontrivial rotations of the cube which leave fixed a pair of opposite edges are determined by the twelve unit quaternions

$$\pm\tfrac{1}{2}i\sqrt{2} \pm \tfrac{1}{2}j\sqrt{2},\ \pm\tfrac{1}{2}j\sqrt{2} \pm \tfrac{1}{2}k\sqrt{2},\ \pm\tfrac{1}{2}k\sqrt{2} \pm \tfrac{1}{2}i\sqrt{2}.$$

The three rotations of the cube which leave fixed a pair of centers of opposite faces and which turn the cube one–half of a complete revolution are represented by the six unit quaternions

$$\pm i,\ \pm j,\ \pm k.$$

The six rotations of the cube which leave fixed a pair of centers of opposite faces and which turn the cube through one–quarter or three–quarters of a complete revolution are represented by the twelve unit quaternions

$$\pm\tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}i\sqrt{2},\ \pm\tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}j\sqrt{2},\ \pm\tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}k\sqrt{2}.$$

The trivial rotation of the cube is represented by the self–conjugate unit quaternions

$$1, -1.$$

The quaternions

$$t + ix + jy + kz$$

whose coefficients x, y, z, and t are rational numbers form a field whose properties resemble the properties of the field of rational numbers. The units of the field are twenty–four of the forty–eight unit quaternions which determine symmetries of a cube. An element of the field is said to be integral if it is a finite sum of units. Sums, products, and conjugates of integral elements of the field are integral. A quaternion is integral if, and only if, its coordinates are all integers or all halves of odd integers. The self–conjugate integral elements of the field are the integers. If ω is a nonzero integral element of the field, then $\omega^-\omega$ is a positive integer. The number of representations

$$n = \omega^-\omega$$

of a positive integer n with ω an integral element of the field is twenty–four times the sum of the odd divisors of n. The proof is an application of an Euclidean algorithm for integral elements of the field. If α is an integral element of the field and if β is a nonzero integral element of the field, then

$$\alpha = \beta\gamma + \delta$$

for integral elements γ and δ of the field such that

$$\delta^-\delta < \beta^-\beta.$$

A determination of the ideals of the ring of integral elements of the field is an application of the Euclidean algorithm. If a right ideal contains a nonzero element, then a least positive integer n exists which admits a representation

$$n = \omega^-\omega$$

with ω in the ideal. Every element of the ideal is then a product

$$\omega\xi$$

with ξ an integral element of the field.

The fundamental nature of rational numbers was observed by Pythagoras in the fifth century BCE. Other numbers are constructed by approximation procedures which are clarified by using the concept of convexity for a set of rational numbers. Nonnegative rational numbers appear as finite sums of squares of rational numbers. A sum of two nonnegative rational numbers is zero only if both rational numbers are zero. A set of rational numbers is said to be convex if it contains

$$(1 - t)a + tb$$

whenever it contains a and b, and t is a nonnegative rational number such that $1 - t$ is nonnegative. Intersections of convex sets are convex. The closure of a convex set is the set of rational numbers a such that

$$(1 - t)a + tb$$

belongs to the convex set for some element b of the set whenever t is a nonnegative number such that $1 - t$ is nonnegative and nonzero. The closure of a convex set is a convex set which contains the given set and which is equal to its closure. A convex set is said to be open if it is disjoint from the closure of every disjoint convex set. A finite intersection of open convex sets is an open convex set. A set of rational numbers is said to be open if it is a union of convex open sets. Unions of open sets are open. Finite intersections of open sets are open. A set of rational numbers is said to be closed if it is the complement of an open set. An intersection of closed sets is closed. A finite union of closed sets is closed. If a and b are distinct rational numbers, then disjoint open set A and B exist such that a is contained in A and b is contained in B.

The rational numbers are then a Hausdorff space with a topology which is compatible with addition and multiplication. Addition and multiplication are continuous as transformations of the Cartesian product of the rational numbers with itself into the rational numbers. This information is sufficient for the usual construction of the real numbers as a completion of the rational numbers. The real numbers are a Hausdorff space with a definition of addition and multiplication as continuous transformations of the Cartesian product of the space with itself into the space. The addition and multiplication of rational numbers is the restriction of the addition and multiplication of real numbers. The initial topology of the rational numbers is the subspace topology of the real numbers. The rational numbers are dense in the real numbers in this topology. If a and b are real numbers, the closure of the set of real numbers of the form

$$(1 - t)a + tb$$

with t a nonnegative rational number such that $1-t$ is rational is a compact set. Every real number belongs to an open set whose closure is compact. These properties of real numbers underlie the usual construction of Lebesgue measure which is used for the definition of the Fourier transformation for the real numbers.

The real numbers are the completion of the rational numbers which appears in the deterministic formulation of mechanics due to Newton. Another completion is required for the uncertain determination of position of a moving object having symmetries of a cube. A completion of the ring of integers is first made. The quotient ring of the integers modulo a nontrivial proper ideal is a finite ring whose discrete topology is the unique topology compatible with addition and multiplication. The adic topology of the integers is the least topology with respect to which all homomorphisms into finite quotient rings are continuous. Addition and multiplication are continuous as transformations of the Cartesian product of the ring of integers with itself into the ring when the ring is given the adic topology. The adic topology of the rational numbers is constructed from the adic topology of the integers. The integers are an open subset of the rational numbers which has the subspace topology of the rational numbers when the rational numbers are given the adic topology. Multiplication by a positive number acts as a homeomorphism of the adic topology of the rational numbers. Addition is continuous as a transformation of the Cartesian product of the ring of rational numbers with itself into the ring when the rational numbers are given the adic topology.

The adic numbers are constructed as the completion of the rational numbers in the adic topology. The adic numbers are a Hausdorff space with a definition of addition and multiplication as continuous transformations of the Cartesian product of the space with itself into the space. The addition and multiplication of rational numbers is the restriction of the addition and multiplication of adic numbers. The adic topology of the rational numbers is the subspace topology of the adic topology of the adic numbers. The rational numbers are dense in the adic numbers in the adic topology. The integral adic numbers are the adic numbers which belong to the closure of the integers in the adic topology. The integral adic numbers form a compact subring of the adic numbers which is a neighborhood of the origin for the adic topology. These properties of the adic numbers underlie the construction of a nonnegative measure which is used for the definition of the Fourier transformation for the adic numbers.

The adic topology is created by the incomplete perception of the motion of a symmetric object. Prime numbers appear in the decomposition of received information into basic components. A completion of the rational numbers is made for every prime p. A completion of the integers is first made. A nontrivial ideal of the integers is generated by a positive integer r and consists of the integers which are divisible by r. The quotient ring is the ring of integers modulo r, which has r elements. The adic topology of the integers applies the homomorphism of the integers onto the integers modulo r for every positive integer r. The p–adic topology applies the homomorphism only when r is a power of the prime p. The p–adic topology of the integers is the least topology with respect to which the homomorphism onto the integers modulo r is continuous whenever r is a power of p. Addition and multiplication are continuous as transformations of the Cartesian product of the ring of integers with itself into the ring when the ring is given the p–adic topology. The p–adic topology of the rational numbers is constructed from the p–adic topology of the integers. The integers are an open subset of the rational numbers which has the subspace topology of the rational numbers when the rational numbers are given the p–adic topology. Multiplication by a positive integer acts as a homeomorphism of the p–adic topology of the rational numbers. Addition is continuous as a transformations of the Cartesian product of the ring of rational numbers with itself into the ring when the rational numbers are given the p–adic topology.

The p–adic numbers are a Hausdorff space with a definition of addition and multiplication as continuous transformatons of the Cartesian product of the space with itself into the space. The addition and multiplication of rational numbers is the restriction of the addition and multiplication of p–adic numbers. The p–adic topology of the rational numbers is the subspace topology of the p–adic topology of the p–adic numbers. The rational numbers are dense in the p–adic numbers in the p–adic topology. The integral p–adic numbers form a compact subring of the p–adic numbers which is a neighborhood of the origin for the p–adic topology. These properties of the p–adic numbers permit the construction of a nonnegative measure which is used for the definition of the Fourier transformation for the p–adic numbers.

The significance of a prime p among positive integers is that the ideal which it generates in the integers is maximal among proper ideals. It follows that the product of nonzero integral p–adic numbers is nonzero. The integral p–adic numbers are then an integral domain whose ring of quotients is the field of p–adic numbers. The properties of the p–adic numbers as a commutative locally compact field create an analogy with the field of real numbers which guides the construction of a function theory for the p–adic numbers. The algebraic interpretation of the p–adic numbers is supplemented by their geometric interpretation as a line which contains rational points as does the real line. The p–adic line can be as relevant as the real line for the observation of motion of a symmetric object whose position in space is undetermined.

The adelic line is a quotient space of the Cartesian product of the real line and the adic line. An element ξ of the Cartesian product space has an Euclidean component ξ_+, which is a real number, and an adic component ξ_-, which is an adic number. A rational number is at once an element of the real line and an element of the adic line. Elements ξ and η of the Cartesian product space are considered equivalent if the identities

$$\eta_+ = \eta_+ + t$$

and

$$\eta_- = \xi_- - t$$

hold for a rational number t. The sum of elements α and β of the adelic line is the element

$$\gamma = \alpha + \beta$$

of the adelic line whose Euclidean component

$$\gamma_+ = \alpha_+ + \beta_+$$

is the sum of the Euclidean components of α and β and whose adic component

$$\gamma_- = \alpha_- + \beta_-$$

is the sum of the adic components of α and β. The product

$$t\xi = \eta = \xi t$$

of a rational number t and an element ξ of the adelic line is the element η of the adelic line whose Euclidean component

$$t\xi_+ = \eta_+ = \xi_+ t$$

is the product of t and the Euclidean component of ξ and whose adic component

$$t\xi_- = \eta_- = \xi_- t$$

is the product of t and the adic component of ξ. These definitions are independent of choices of representatives in equivalence classes.

The topology of the adic line is the quotient of the Cartesian product of the topologies of the Euclidean and adic lines. A computation of the quotient topology is made by the construction of a fundamental region for the equivalence relation on the Cartesian product space. The fundamental region is the Cartesian product of the open interval $(-1/2, 1/2)$ of the Euclidean line and the set of integral elements of the adic line. The fundamental region is an open subset of the Cartesian product space such that every element of the space is equivalent to an element of the closure of the region, which is the Cartesian product of the closed interval $[-1/2, 1/2]$ and the set of integral elements of the adic line. Equivalent elements of the fundamental region are equal. Distinct equivalent elements of the closure of the fundamental region occur in pairs whose Euclidean components are one–half and minus one–half and whose adic components differ by one. The fundamental region is a neighborhood of the origin whose closure is compact. For every rational number t the transformation which takes (ξ_+, ξ_-) into $(\xi_+ + t, \xi_- - t)$ maps the fundamental region onto an open set which is a neighborhood of an element $(t, -t)$ equivalent to the origin. The elements of the Cartesian product space which are equivalent to the origin form a discrete subgroup whose quotient group is a compact Hausdorff space.

A dense subset of the adelic line consists of the elements which are represented by an element of the Cartesian product space with rational adic component. Since these elements are represented by pairs with adic component equal to zero, the adelic line is a completion of the Euclidean line in a topology which is compatible with additive structure. The topology is the weakest topology with respect to which the function

$$\exp(2\pi i t\xi_+)$$

is a continuous function of real numbers ξ_+ when the unit circle is given its Euclidean topology. The function

$$\exp(2\pi i\xi)$$

of elements ξ of the adelic line is defined as

$$\exp(2\pi i\xi_+)$$

when the adic component of ξ_- is equal to zero, and is otherwise defined by continuity. The identity

$$\exp(2\pi i\xi + 2\pi i\eta) = \exp(2\pi i\xi)\exp(2\pi i\eta)$$

holds for all elements ξ and η of the adelic line.

Addition is continuous as a transformation of the Cartesian product of the adelic line with itself into the adelic line when the adelic line is given its adelic topology. Haar measure for the adelic line is an essentially unique nonnegative measure on the Borel subsets of the adelic line for which a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the adelic line. Multiplication by t is a measure preserving transformation for every nonzero rational number t. Haar measure is normalized so that the adelic line has measure one. The measure is computed on the fundamental region as the Cartesian product of Lebesgue measure on the interval $(-1/2, 1/2)$ and a normalization of Haar measure for the space of integral adic numbers. The taking of residue classes modulo r map Haar measure for the integral adic numbers into the measure which assigns equal mass $1/r$ to each of the r integers modulo r.

The requirements of Fourier analysis discovered by Fourier are met by an integration theory which is due to Henri Lebesgue (1875–1941) in the essential case of the real line. Fourier analysis is analogous for other locally compact abelian groups. The functions

$$\exp(2\pi i t\xi)$$

of ξ in the adelic line which are defined by rational numbers t form a complete orthonormal set in the space of square integrable functions with respect to Haar measure for the adelic line.

Properties of functions on the adelic line result from applications of the Poisson summation formula for the Cartesian product of the Euclidean line and the adic line. Lebesgue measure is used in the definition of the Fourier transformation for the Euclidean line. The Fourier transform of a square integrable function $f(\xi)$ of real ξ is a square integrable function $g(\xi)$ of real ξ which is defined formally as an integral

$$g(\eta) = \int \exp(2\pi i\eta\xi) f(\xi) d\xi$$

with respect to Lebesgue measure. The integral is applied as the definition when it is absolutely convergent, in which case the identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

is satisfied with integration with respect to Lebesgue measure. The transformation is otherwise defined so as to maintain the identity. The function $f(\xi)$ of ξ is then recovered as the inverse Fourier integral

$$f(\eta) = \int \exp(-2\pi i\eta\xi) g(\xi) d\xi$$

which has a similar interpretation.

The Fourier transformation for the adic line is defined using the normalization of Haar measure for the adic line which assigns measure one to the set of integral adic numbers. The function

$$\exp(2\pi i\xi)$$

of rational numbers ξ is continuous with respect to the adic topology of the rational numbers. The function has a unique continuous extension as a function

$$\exp(2\pi i\xi)$$

of ξ in the adic line. The identity

$$\exp(2\pi i\xi + 2\pi i\eta) = \exp(2\pi i\xi)\exp(2\pi i\eta)$$

holds for all elements ξ and η of the adic line. The Fourier transform of a square integrable function $f(\xi)$ of ξ in the adic line is a square integrable function $g(\xi)$ of ξ in the adic line which is defined formally as the integral

$$g(\eta) = \int \exp(2\pi i\eta\xi)f(\xi)d\xi$$

with respect to Haar measure for the adic line. The integral is applied as the definition when it is absolutely convergent, in which case the identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

is satisfied with integration with respect to Haar measure. The Fourier transformation is otherwise defined so as to maintain the identity. The function $f(\xi)$ of ξ in the adic line is recovered as the inverse Fourier integral

$$f(\eta) = \int \exp(-2\pi i\eta\xi)g(\xi)d\xi$$

which has a similar interpretation.

The Cartesian product measure of Lebesgue measure for the Euclidean line and Haar measure for the adic line is used for the definition of the Fourier transformation for the Cartesian product space of the Euclidean line and the adic line. The Fourier transform of a square integrable function $f(\xi)$ of ξ in the Cartesian product space is a square integrable function $g(\xi)$ of ξ in the Cartesian product space which is defined formally as an integral

$$g(\eta) = \int \exp(2\pi i\eta_+\xi_+ - 2\pi i\eta_-\xi_-)f(\xi)d\xi$$

with respect to the Cartesian product measure. The integral is accepted as the definition when it is absolutely convergent, in which case the identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

is satisfied with integration with respect to the Cartesian product measure. The transformation is otherwise defined so as to preserve the identity. The function $f(\xi)$ of ξ in the Cartesian product space is recovered as the inverse Fourier integral

$$f(\eta) = \int \exp(2\pi i\eta_-\xi_- - 2\pi i\eta_+\xi_+)g(\xi)d\xi$$

which has a similar interpretation.

Poisson summation is a construction of integrable functions on the adelic line from integrable functions on the Cartesian product space. The Poisson sum of a function $f(\xi)$

of ξ in the Cartesian product space is the function $g(\eta)$ of η in the adelic line obtained as a sum

$$g(\eta) = \sum f(\eta + \xi)$$

over the elements ξ of the adelic line which are equivalent to the origin. If $f(\xi)$ is an integrable function of ξ in the Cartesian product space, then the sum is absolutely convergent for almost all η in the adelic line and the resulting function $g(\eta)$ of η in the adelic line is integrable. The inequality

$$\int |g(\eta)| d\eta \leq \int |f(\xi)| d\xi$$

holds with integration on the left over the adelic line and with integration on the right over the Cartesian product space.

The Poisson formula relates a Poisson sum constructed from a square integrable function $f(\xi)$ of ξ in the Cartesian product space and the Poisson sum constructed from the square integrable function $g(\xi)$ of ξ in the Cartesian product space which is its Fourier transform when $f(\xi)$ and $g(\xi)$ are integrable functions of ξ. The functions $f(\xi)$ and $g(\xi)$ are then continuous functions of ξ. The Poisson sums are continuous functions on the adelic line. Since the adelic line is a compact Hausdorff space, the Poisson sums are square integrable functions on the adelic line. The Fourier transform of the square integrable function

$$\exp(-2\pi i \beta \xi) f(\xi + \alpha)$$

of ξ in the Cartesian product space is the square integrable function

$$\exp(2\pi i \alpha \beta) \exp(-2\pi i \alpha \xi) g(\xi - \beta)$$

of ξ in the Cartesian product space for all elements α and β of the Cartesian product space.

The Poisson formula states that the Poisson sums of these Fourier transforms have equal values at the origin. The Poisson sums are continuous functions of β in the Cartesian product space whose values depend only on the equivalence class of β. The Poisson formula states that the sums

$$\sum \exp(-2\pi i \xi \eta) f(\xi + \alpha)$$

and

$$\sum \exp(2\pi i \alpha \eta - 2\pi i \alpha \xi) g(\xi - \beta)$$

over the elements ξ of the Cartesian product space which are equivialent to the origin define equal functions of η in the adelic line. The Poisson formula is proved by showing that the integrals

$$\int \exp(2\pi i \lambda \eta) \sum \exp(-2\pi i \xi \eta) f(\xi + \alpha) d\eta$$

and

$$\int \exp(2\pi i \lambda \eta) \sum \exp(2\pi i \alpha \eta - 2\pi i \alpha \xi) g(\xi - \beta) d\eta$$

over the adelic line are equal for every element λ of the adelic line which is equivalent to zero. Since

$$\exp(2\pi i \lambda \eta) = \exp(2\pi i t \eta)$$

when t is the rational number equal to the Euclidean component of λ, the functions

$$\exp(2\pi i\lambda\eta)$$

form a complete orthonormal set in the space of square integrable functions of η in the adelic line. Interchanges of summation and integration are justified by absolute convergence. It needs to be shown that the sums

$$\sum f(\xi+\alpha)\int \exp(2\pi i\lambda\eta - 2\pi i\xi\eta)d\eta$$

and

$$\sum \int \exp(2\pi i(\lambda+\alpha)(\eta-\xi))g(\xi-\eta)d\eta$$

are equal. Since the integral

$$\int \exp(2\pi i\lambda\eta - 2\pi i\xi\eta)d\eta$$

over the adelic line is equal to one when ξ is equal to λ and is equal to zero otherwise, the first sum is equal to

$$f(\lambda+\alpha).$$

Since the second sum is equal to the integral

$$\int \exp(2\pi i(\lambda+\alpha)(\eta-\xi))g(\xi-\eta)d\eta$$

over the Cartesian product space, it is equal to

$$f(\lambda+\alpha)$$

by the Fourier inversion formula.

A spherical harmonic of order ν is a homogeneous polynomial of degree ν in x, y, z which is a solution of the Laplace equation. The polynomial is treated as a function $f(\xi)$ of the quaternion variable

$$\xi = t + ix + jy + kz.$$

The spherical harmonics of order ν form a vector space of dimension $1+2\nu$ over the complex numbers. An irreducible representation of the rotation group on three–dimensional space is defined by taking a function $f(\xi)$ of quaternions ξ into the function $f(\omega^-\xi\omega)$ of quaternions ξ for every unit quaternion ω. The vector space admits an essentially unique scalar product with respect to which the representation is unitary. The representation is fundamental to the quantum mechanical theory of orbital electrons.

A construction of zeta functions is made from spherical harmonics of order ν which are left fixed by the rotations corresponding to integral units. These harmonics are functions $f(\xi)$ of a quaternion variable ξ which satisfy the symmetry condition

$$f(\xi) = f(\omega^-\xi\omega)$$

for every integral unit ω. The spherical harmonics of order ν which satisfy the symmetry condition form a vector space of finite dimension which is acted upon by commuting self–adjoint transformations. The transformation $\Delta(n)$ parametrized by an odd positive integer

n takes a function $f(\xi)$ of a quaternion variable ξ into a function $g(\xi)$ of a quaternion variable ξ defined by the sum

$$24n^\nu g(\xi) = \sum f(\omega^- \xi \omega)$$

over the integral quaternions ω such that

$$n = \omega^- \omega.$$

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all odd positive integers m and n with summation over the common divisors k of m and n. The space of spherical harmonics of order ν which satisfy the symmetry condition admits an orthogonal basis whose elements are eigenfunctions of $\Delta(n)$ for every odd positive integer n. A basic element is an eigenfunction of $\Delta(n)$ for a real eigenvalue $\tau(n)$. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all odd positive integers m and n with summation over the common divisors k of m and n. The zeta function $\zeta(s)$ defined by a basic element is a sum

$$\zeta(s) = \sum \tau(n)n^{-s}$$

over the odd positive integers n which converges in the half–plane $\mathcal{R}s > 1$. The zeta function admits an Euler product and satisfies a functional identity analogous to the Euler product and functional identity of Euler and Dirichlet zeta functions.

Since the history of the Riemann hypothesis is complicated, I will approach it from the proof of the view of my own involvement. The first aim is to account for the choice of the Riemann hypothesis as a research objective. Mathematicians ordinarily choose a research career under the guidance of a professor who supervises a doctoral thesis. But I came to mathematics at an earlier age under the influence of someone not associated with a university. I would not have had such a teacher had it not been for unusual circumstances resulting from the Second World War. I was born in Paris in 1932 and attended school there until 1941 when the German occupation compelled the departure of my mother and her three children to the United States. Since my father remained in France, my maternal grandparents assumed a responsibility that would normally fall to parents.

When the United States entered the war six months later, I was safe with my mother and sisters in a seashore cottage at Rehoboth Beach, Delaware. My adaptation to English as a primary language was eased by the summer visits made to my grandparents in my childhood. My earliest mathematical experience was obtained solving cryptograms in the Philadelphia Inquirer. The mystery of wartime secrecy stimulated logical thought. Deciphering coded messages was part of the general effort for survival.

My progress in school was sufficient for me to omit seventh grade. When I was twelve, I entered Saint Andrew's School, Middletown, Delaware. Since the cottage at Rehoboth was then sold, the house of my grandparents in Wilmington, Delaware, was home during vacations. My grandfather, Ellice Mc Donald, was a former surgeon and university lecturer who had turned to research. Research careers had recently been made possible by the Rockefeller Institute. My grandfather found an alternative at the Franklin Institute by founding the Biochemical Research Foundation, of which he was director. The foundation moved from its original quarters in Philadelphia to new quarters off the campus of the University of Delaware as the United States entered the war.

I took my studies more seriously than other students did because my grandfather convinced me that it was important to do so. I was stimulated by elementary algebra, which I studied in the third form. In the following summer vacation I solved the problems of an exercise book of intermediate algebra and was advanced to plane geometry for the fourth form. When I was home on vacations, I accompanied my grandfather to the Biochemical Research Foundation. And I caddied for him on Sunday mornings when he played golf with Irénée du Pont, the former president of the du Pont Company who supplied the funds for the Biochemical Research Foundation.

Mr. du Pont always drank a glass of rhum and orange juice in the clubhouse after playing golf. One morning he showed an unexpected interest in my mathematical education by posing a problem: Find positive integers a, b, and c such that

$$a^3 + b^3 = 22c^3.$$

Since the problem was more interesting than plane geometry, I spent the fourth form year solving it. For this purpose I had access to the libraries of Saint Andrew's School, the Biochemical Research Foundation, and the University of Delaware. With the help of these sources I was able to learn the representation theory of positive integers in the form $a^2 - ab + b^2$ for integers a and b. This information is a prerequisite to a solution of the problem, which I have unfortunately lost. It was an achievement comparable to my doctoral thesis written ten years later. The result was difficult to check because the numbers obtained had five and six digits. Cubing them was beyond the capacity of the Marchant calculators available at the Biochemical Research Foundation. Mr. du Pont conceded the correctness of the solution but never revealed the source of the problem. This variant of the Fermat problem originates with Lagrange, who states it however with 10 instead of 22.

Another significant mathematical experience occurred in my fifth form year. I learned from a graduate text of the existence of a generalization of the factorial called the gamma function. In the course of the year I rediscovered the Euler product for the function without any training in complex analysis. A good understanding of the calculus was sufficient. The gamma function remained as an interest decisive in the proof of the Riemann hypothesis.

Since my grandfather was pleased with my mathematical progress, he decided that I should have a college education. In the fall semester 1949 I entered the Massachusetts Institute of Technology, the university at which Mr. du Pont had been an undergraduate. I took my first mathematics course with George Thomas as he was writing his calculus text. I used his manuscript to prepare the remaining three semesters of the calculus, which I then disposed of in proficiency examinations. I was able to take a graduate course in linear algebra from Witold Hurewicz in the second semester. The text on *Modern Algebra* by Garrett Birkhoff and Saunders Mac Lane was familiar as it had been in the library of the Biochemical Research Foundation. During the summer I worked through the recently published *Lectures on Classical Differential Geometry* by Dirk Struik. Walter Rudin taught me a course in my sophomore year on the *Principles of Mathematical Analysis*. The aim of my undergraduate education was however not to prepare for a career in mathematics, but to acquire knowledge of value in applications to science. I obtained an undergraduate degree in chemistry as well as mathematics. I learned about scattering theory in physics courses and about linear systems in engineering courses. But my talents lie in theory rather than applications. The eventual decision to become a mathematician was a rupture with the scientific aims of my grandfather.

When I decided on graduate school in mathematics, I was discouraged by my mathematical advisors from applying to the prestigious universities Harvard and Princeton because of an insufficient concentration on courses in mathematics. With the backing of George Thomas I received a teaching assistantship at Cornell University for the fall semester 1953. The terrain lost as an undergraduate was recovered in the first graduate year. My grandfather died in the middle of the second year as I was taking qualifying examinations for the doctoral program.

I returned to number theory on passing the examinations. Preparation for them included lecture notes of Emil Artin and Emma Nöther on *Moderne Algebra* taken by Bartel van der Waerden. Three treatises by Edward Titchmarsh then determined the direction of my efforts: *Introduction to the Theory of Fourier Integrals, Eigenfunction Expansions Associated with Second-Order Differential Equations,* and *The Theory of the Riemann Zeta-Function*. The Riemann hypothesis is a unifying theme of these volumes which became the ultimate goal of my research.

The attack on the Riemann hypothesis begins in my doctoral thesis, which concerns a problem in Fourier analysis, due to Arne Beurling, which was posed by Harry Pollard to Wolfgang Fuchs. An axiomatization in Hilbert space was made in postdoctoral work. Assume that a nonnegative measure μ is given on the Borel subsets of the real line with respect to which all polynomials are square integrable. Determine the closure of the polynomials in $L^2(\mu)$ when the closure is not the whole space. A Hilbert space of entire functions is obtained which has these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z - w^-)/(z - w)$ belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional is defined on the space by taking $F(z)$ into $F(w)$ for every nonreal number w.

(H3) The function $F^*(z) = F(z^-)^-$ belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

Examples of spaces with these properties appear in the Colloquium Publication of Raymond Paley and Norbert Wiener, *Fourier Transforms in the Complex Domain,* American Mathematical Society, 1934. If a positive number a is given, the Paley-Wiener space of index a is the set of entire functions $F(z)$ of the form

$$2\pi F(z) = \int_{-a}^{a} f(t) \exp(itz) dt$$

with a finite integral

$$2\pi \|F\|^2 = \int_{-a}^{a} |f(t)|^2 dt.$$

The elements of the space are the entire functions of exponential type at most a which are square integrable on the real axis. The norm of the space is computable in several ways in terms of function values on the real axis since the identity

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = (\pi/a) \sum_{-\infty}^{+\infty} |F(n\pi/a)|^2$$

holds for every element $F(z)$ of the space. The identity was observed in a related context in 1814 by Carl Friederich Gauss, *Methodus nova integralium valores per approximationen* *inveniendi*, *Werke,* Königliche Gesellschaft der Wissenschaften, Göttingen, 1886, volume 3, pp. 163–196.

A generalization of Gaussian quadrature applies in Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3). The structure theory for such a space is related to the theory of entire functions $E(z)$ which satisfy the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for $y > 0$. Write

$$E(z) = A(z) - iB(z)$$

where $A(z)$ and $B(z)$ are entire functions which are real for real z and

$$K(w, z) = \frac{B(z)A(w)^- - A(z)B(w)^-}{\pi(z - w^-)}.$$

Then the set of entire functions $F(z)$ such that the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt$$

is finite and such that the inequality

$$|F(z)|^2 \leq \|F\|^2 K(z, z)$$

holds for all complex numbers z, is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). The function $K(w, z)$ of z acts as a reproducing kernel function for function values at w since it is the unique element of the space which satisfies the identity

$$F(w) = \langle F(t), K(w, t)\rangle$$

for every element $F(z)$ of the space. A Hilbert space, whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is isometrically equal to a space $\mathcal{H}(E)$.

The Paley-Wiener space of index a is obtained when

$$E(z) = \exp(-iaz)$$

in which case

$$A(z) = \cos(az)$$

and

$$B(z) = \sin(az).$$

The definition of the norm of the space $\mathcal{H}(E)$ simplifies since $E(z)$ has modulus one on the real axis.

Multiplication by z in a space $\mathcal{H}(E)$ is the transformation which takes $F(z)$ into $G(z)$ whenever $F(z)$ and $G(z)$ are elements of the space such that

$$G(z) = z\,F(z).$$

Multiplication by z need not be densely defined in a space $\mathcal{H}(E)$, but its domain is nearly dense. The orthogonal complement of the domain consists of those elements of the space which are of the form

$$A(z)u + B(z)v$$

for complex numbers u and v. Since such numbers satisfy the identity

$$v^- u = u^- v,$$

the orthogonal complement of the domain has dimension zero or one.

A generalization of Gaussian quadrature applies in a space $\mathcal{H}(E)$. A phase function associated with $E(z)$ is a continuous function $\phi(x)$ of real x with real values such that the product

$$E(x) \exp[i\phi(x)]$$

has real values. Such a function exists and is unique within an added integer multiple of π. The function is differentiable and has positive derivative everywhere. If α is a given real number, the inequality

$$\|F\|^2_{\mathcal{H}(E)} \leq \sum |F(t)/E(t)|^2 \pi/\phi'(t)$$

holds for every element $F(z)$ of the space with summation over the real numbers t such that $\phi(t)$ is congruent to α modulo π. Equality holds for every element $F(z)$ of the space when the function

$$E(z) \exp(i\alpha) - E^*(z) \exp(-i\alpha)$$

does not belong to the space. At most one real number α modulo π exists such that the function belongs to the space. The function then spans the orthogonal complement of the domain of multiplication by z in the space. Equality holds for every element $F(z)$ of the closure of the domain of multiplication by z in the space. The quadrature identity of Fourier analysis is recovered when

$$E(z) = \exp(-iaz)$$

for a positive number a, in which case the identity

$$\phi(x) = ax$$

is satisfied.

The quadrature identity is relevant to the Riemann hypothesis as the conjecture that the zeros of certain entire functions are real. The significance of the Riemann hypothesis is that the quadrature identity applies in a context relevant to the asymptotic distribution of prime numbers. The theory of Hilbert spaces of entire functions is an interpretation of the work of Thomas Stieltjes, *Recherches sur les fractions continues*, Annales de la Faculté Scientifique de Toulouse 8 (1894), 1-122, and 9 (1895), 1-47. His analytic theory of continued fractions is reformulated in the theory of Hilbert spaces of entire functions as a factorization theory for matrix-valued analytic functions.

The resemblance of the theory of Hilbert spaces of entire functions to Fourier analysis is extensive and substantial. A generalization of Fourier analysis is associated with every nontrivial Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). Every such space is embedded in a family of such spaces similar to the Paley-Wiener spaces.

A partial ordering of Hilbert spaces of entire functions is implicit in the construction of such families. A space $\mathcal{H}(E(a))$ with index a is considered less than or equal to a space $\mathcal{H}(E(b))$ with index b if the ratio

$$E(a,z)/E(b,z)$$

has no real zeros, if the space with index a is contained contractively in the space with index b, and if the inclusion is isometric on the domain of multiplication by z in the space with index a. A fundamental theorem states that such Hilbert spaces of entire functions appear in totally ordered families. If a space $\mathcal{H}(E(a))$ with index a and a space $\mathcal{H}(E(b))$ with index b are less than or equal to a space $\mathcal{H}(E(c))$ with index c, then either the space with index a is less than or equal to the space with index b or the space with index b is less than or equal to the space with index a. A nontrivial Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is a member of a maximal totally ordered family of such spaces. The members of the family are indexed by real numbers in such a way that the space with index a is less than or equal to the space with index b when a is less than or equal to b. Every member of the family with index b is then the least upper bound of members of the family with index a less than b. Every member of the family with index a is also the greatest lower bound of the members of the family with index b greater than a.

These results, which were obtained in the postdoctoral years 1957-1962, are published in *Hilbert Spaces of Entire Functions,* Prentice-Hall, 1968. The structure of mathematical journals creates the impression that mathematics is fragmented into unrelated disciplines. The underlying unity of mathematics is however maintained by problems which span these disciplines. A selection of such problems was presented by David Hilbert to the International Congress of Mathematicians which was held in 1900 in Paris: *Mathematical Problems*, Bulletin of the American Mathematical Society 8 (1902), 437–479. The Riemann hypothesis is listed as an important link between algebra and analysis.

The analytic aspects of the asymptotic behavior of prime numbers originate in the gamma function, discovered in 1730 by Leonard Euler, *De progressionibus transcendentalibus seu quarum termini generales algebraice dari nequeunt*, Opera Omnia I (14), 1–24. His discovery of the Euler product for the classical zeta function was made in 1737, *Variae observationes circa series infinitas*, Opera Omnia I (14), 216–244. A substantial evolution in the theory of the gamma function is required for the functional identity which Euler discovered for the classical zeta function in 1761: *Remarques sur un beau rapport entre les séries de puissances tant directes que réciproques*, Opera Omnia I (15), 70–90. The Riemann hypothesis for the classical zeta function was stated by Bernhard Riemann in 1859. No motivation for the conjecture was published by Riemann although he is known to have made calculations of zeros which were later duplicated by Jean-Pierre Gram, *Note sur les zéros de la fonction de Riemann*, Acta Mathematica 27 (1903), 289–305.

The classical motivation for the Riemann hypothesis is attributed to Nikolai Sonine, *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries*, Mathematische Annalen 16 (1880), 1-80. Remarkable examples of functions related to zeta functions are presented for which the analogue of the Riemann hypothesis is true. A spectral theory involving the gamma function is derived from properties of the Hankel transformation of order zero. Sonine observes that a square integrable function and its Hankel transform of order zero can vanish in a neighborhood of the origin without vanishing identically. An axiomatic treatment in the theory of Hilbert spaces of entire functions was given by the author, *Self-reciprocal functions*, Journal of Mathematical Analysis and

Applications 9 (1964), 433-455. A parametrization is made of all square integrable functions which vanish in an interval containing the origin and whose Hankel transform of order zero vanishes in the same interval. A derivation of the expansion from the examples given by Sonine was made by Virginia Rovnyak in her thesis, *Self-reciprocal functions*, Duke Mathematical Journal 33 (1966), 363-378. A generalization of the expansion for the Hankel transformation of integer order was made by James and Virginia Rovnyak, *Self-reciprocal functions for the Hankel transformation of integer order*, Duke Mathematical Journal 34 (1967), 771-785. These results are less complete than those for the Hankel transformation of order zero.

The Riemann hypothesis as a research objective created a career obstacle since the relevance of the theory of Hilbert spaces of entire functions could not be established. When tenured positions were unavailable in the vicinity of Philadelphia, I accepted in 1962 the offer of an associate professorship on the Lafayette campus of Purdue University. Promotion to professor was immediate. Philadelphia retained its significance as an educational and research center because vacations could be spent there. The city supplied students who came to Lafayette for doctoral and postdoctoral work. A construction of Hilbert spaces of entire functions associated with Dirichlet zeta functions was made during this time.

If ρ is a given positive integer, a character modulo ρ is a function $\chi(n)$ of integers n, which is periodic of period ρ, which satisfies the identity

$$\chi(mn) = \chi(m)\chi(n)$$

for all integers m and n, which has nonzero values at integers relatively prime to ρ, and which vanishes otherwise. A character is an even or an odd function. A character χ modulo ρ is said to be primitive modulo ρ if no character modulo a proper divisor of ρ exists which agrees with χ at integers which are relatively prime to ρ. A character is said to be real if it has real values. The principal character modulo ρ is the unique character modulo ρ whose only nonzero value is one. The character is primitive when ρ is one.

The Dirichlet zeta function associated with a character χ modulo ρ is defined by

$$\zeta(s) = \sum \chi(n) n^{-s}$$

with summation over the positive integers n. The series is absolutely convergent when $\mathcal{R}s > 1$ and represents an analytic function of s in the half-plane. The classical zeta function, which was discovered by Euler, is the Dirichlet zeta function when χ is the principal character modulo one. The function has an analytic extension to the complex plane except for a simple pole at $s = 1$. The Dirichlet zeta function has an analytic extension to the complex plane when χ is not a principal character.

The Dirichlet zeta function $\zeta(s)$ satisfies a functional identity when χ is a primitive real character modulo ρ. The functions

$$(\rho/\pi)^{\frac{1}{2}\nu + \frac{1}{2}s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s) \zeta(s)$$

and

$$(\rho/\pi)^{\frac{1}{2}\nu + \frac{1}{2} - \frac{1}{2}s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}s) \zeta(1 - s)$$

are linearly dependent with $\nu = 0$ when χ is even and $\nu = 1$ when χ is odd. The functions are entire when χ is not the principal character. When χ is the principal character, the functions are equal with simple poles at $s = 0$ and $s = 1$.

The Dirichlet zeta function $\zeta(s)$ has no zeros in the half-plane $\mathcal{R}s > 1$ since the Euler product

$$\zeta(s)^{-1} = \Pi(1 - \chi(p)p^{-s})$$

converges in the half-plane. The product is taken over the primes p which are not divisors of ρ. A less obvious consequence of convergence in the half-plane, due to Hadamard and de la Vallée Poussin, is the absence of zeros in the closure of the half-plane. The functional identity reduces the determination of zeros to the critical strip $0 < \mathcal{R}s < 1$. These zeros are symmetric about the critical line $\mathcal{R}s = \frac{1}{2}$ by the functional identity. The Riemann hypothesis is the conjecture that the zeros lie on the critical line. Although Riemann stated the conjecture for the classical zeta function, it is applied to Dirichlet zeta functions associated with real characters. Simplicity of zeros is a later strengthening of the conjecture.

David Hilbert is said to have assigned the Riemann hypothesis as a thesis problem to his student Erhard Schmidt. The interest of Hilbert in the Riemann hypothesis is attested by his 1900 Congress address. The direction of his interests is further indicated by a series of publications, *Grundzüge einer allgemeinen Theorie der Integralgleichungen*, Göttinger Nachrichten I (1904), 49–91, II (1904), 213–259, III (1905), 307–338, IV (1906), 157–222, and V (1906), 439–480. Erhard Schmidt also made a contribution to the theory of integral equations, *Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener*, Dissertation, Göttingen, 1905. These results include a spectral theory of self-adjoint transformations with discrete spectrum. Hilbert is said to have proposed the construction of such a transformation whose eigenvalues are zeros of the classical zeta function in the critical strip. Supporting evidence is found in a predoctoral publication by Erhard Schmidt, *Über die Anzahl der Primzahlen unter einer gegebenen Grenze*, Mathematische Annalen 57 (1903), 195–204.

The Hilbert strengthening of the Riemann hypothesis is interpreted as the construction of a space $\mathcal{H}(E)$ which is related to the Dirichlet zeta function $\zeta(s)$ associated with a primitive real character χ modulo ρ. The substitution $s = \frac{1}{2} - iz$ converts the function

$$(\rho/\pi)^{\frac{1}{2}\nu + \frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s)\zeta(s)$$

of s into an entire function of z when χ is a nonprincipal character. The function needs to be multiplied by $s(1-s)$ for the same conclusion when χ is the principal character. The functional identity states that the entire function of z is real for real z. The Riemann hypothesis is the conjecture that the function has only real simple zeros. The Hilbert conjecture is interpreted as the existence of a space $\mathcal{H}(E)$ such that the entire function which is real for real z coincides with $A(z)$ in the decomposition

$$E(z) = A(z) - i\,B(z).$$

The Hilbert-Schmidt spectral theory of self-adjoint transformations is an application of the Hadamard factorization of entire functions, which was later axiomatized by Georg Pólya. An entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if it satisfies the inequality

$$|E(x - iy)| \leq |E(x + iy)|$$

for $y > 0$, and if $|E(x + iy)|$ is a nondecreasing function of positive y for every real number x. A space $\mathcal{H}(E)$ then exists when the functions $E(z)$ and $E^*(z)$ are linearly independent.

Multiplication by z in the space admits a self-adjoint extension whose spectrum is contained in the zeros of $A(z)$. The existence of the extension is an application of Gaussian quadrature. The Hilbert-Schmidt spectral theory applies because the zeros t_n of $A(z)$ satisfy a convergence condition. The sum

$$\sum 1/(1+t_n^2)$$

is finite. The Hadamard factorization asserts the existence of sufficiently many zeros for a product representation of $A(z)$. The spectral theory applied in the proof of the Riemann hypothesis is a special case of the Hilbert–Schmidt theory.

The construction of Hilbert spaces of entire functions associated with Dirichlet zeta functions is also an application of the representation theory of the group of matrices of rank two with real entries and determinant one. The modular group is the subgroup formed by the matrices with integer entries. If ρ is a given positive integer, the corresponding Hecke subgroup of the modular group consists of those matrices whose subdiagonal entry is divisible by ρ. A corresponding Hilbert space is constructed for every primitive real character χ modulo ρ. The Hilbert space consists of (equivalence classes of) measurable functions $f(z)$ of z in the upper half-plane such that the identity

$$f(z) = \frac{\chi(D)}{(Cz+D)^{1+\nu}} f(\frac{Az+B}{Cz+D})$$

holds for every element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the Hecke subgroup of the modular group and such that the integral

$$\|f\|^2 = \iint |f(x+iy)|^2 y^{\nu-1} dx\, dy$$

is finite with integration over a fundamental region for the group.

The Laplace-Beltrami operator is a self-adjoint transformation in the space defined formally by taking $f(z)$ into

$$-(z^- - z)^2 \frac{\partial^2 f}{\partial z^- \partial z} + (1+\nu)(z^- - z)\frac{\partial f}{\partial z}.$$

Formal eigenvectors of the transformation are represented by Eisenstein series

$$\sum \frac{\chi(D)}{(Cz+D)^{1+\nu}} f(\frac{Az+B}{Cz+D})$$

using functions $f(z)$ of z in the upper half-plane which are periodic of period one. Summation is over all lower rows of elements

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the Hecke subgroup. The spectral theory of the self-adjoint transformation permits the construction of a Hilbert space $\mathcal{H}(E)$ which is related to the Dirichlet zeta function $\zeta(s)$ associated with the given character. When $s = 1 - iz$, the function

$$(\rho/\pi)^{\frac{1}{2}\nu+\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s)\zeta(s)$$

is equal to the desired function $E(z)$ if χ is not the principal character.

The construction of Hilbert spaces of entire functions associated with Dirichlet zeta functions appeared as *Modular spaces of entire functions*, Journal of Mathematical Analysis and Applications 44 (1973), 192-205. The spectral theory is an interpretation of the results of Hans Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Mathematische Annalen 121 (1949), 141-183. The spaces do not verify the Hilbert conjecture since the spectral line is not the critical line but the right boundary of the critical strip. No information is obtained about zeros of zeta functions in the critical strip. The spaces can be constructed from the Hadamard factorization without recourse to the spectral theory of the Laplace-Beltrami operator. The spectral theory does however indicate that the spaces are natural to zeta functions. The spaces are a generalization of the spaces of the Sonine theory, which are applied in the construction of spaces from the Maass theory. The Sonine spaces solve a problem of parametrization of square integrable functions which vanish in an interval containing the origin and whose Hankel transform of a given order vanishes in the same interval. The Maass spaces apply to a similar problem formulated in the proof of the Riemann hypothesis.

Remarks on the Hankel transformation are appropriate because of its appearance in the Sonine theory. An axiomatic treatment of the Hankel transformation of order ν is an elementary application of the theory of Hilbert spaces of entire functions. Assume that ν is a given real number. A space $\mathcal{H}(E)$ is said to be homogeneous of order ν if an isometric transformation of the space onto itself is defined by taking $F(z)$ into $a^{1+\nu}F(az)$ when $0 < a < 1$. The Paley-Wiener spaces are homogeneous of order $-\frac{1}{2}$. Related spaces exist when $\nu > -1$, in which case the norm of the space is defined by

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)|^2 |t|^{2\nu+1} dt.$$

The spaces appear in the theory of the Hankel transformation of order ν, which is defined using the Bessel function

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{\nu+2n}}{\Gamma(1+n)\Gamma(1+\nu+n)}.$$

If $f(x)$ is a square integrable function of positive x, its Hankel transform of order ν is defined by

$$g(x) = \int_0^\infty f(t) J_\nu(xt) \sqrt{xt}\, dt$$

when the integral is absolutely convergent. A square integrable function $g(x)$ of positive x is obtained which satisfies the identity

$$\int_0^\infty |f(t)|^2 dt = \int_0^\infty |g(t)|^2 dt.$$

The isometric property of the transformation permits its definition on the space $L^2(0,\infty)$. The transformation is its own inverse. A self-reciprocal function is its own Hankel transform. A skew-reciprocal function is minus its own Hankel transform. Every element of the space is the orthogonal sum of a self-reciprocal function and a skew-reciprocal function. A related Hilbert space of entire functions, which is homogeneous of order ν, is obtained for every positive number a. The even elements of the space are the entire functions $F(z)$ such that $x^{-\nu}F(x)$ is the Hankel transform of order ν of a function which vanishes outside of the interval $(0, a)$.

A fundamental example of a self-reciprocal function of order ν is

$$x^{\nu+\frac{1}{2}}\exp(-\tfrac{1}{2}x^2).$$

A construction of Hankel transform pairs is made using the Laplace transformation. Sonine applies the construction to produce Hankel transform pairs which vanish in a given interval containing the origin. The construction of all such pairs is a fundamental problem which admits a solution when $\nu = 0$. Contiguous relations between the Hankel transformation of order ν and the Hankel transformation of order $\nu + 1$ permit a solution when ν is a nonnegative integer.

The Hankel transformation of order minus one-half is the cosine transformation. The Hankel transformation of order one-half is the sine transformation. These transformations are derived from the Fourier transformation for the real numbers under a decomposition which results from inversion about the origin. The Hankel transformation of integer order is derived from the Fourier transformation for the plane under a similar decomposition which results from the action of rotations about the origin. The derivation of the Hankel transformation from the Fourier transformation permits generalizations in which the real numbers are replaced by a locally compact field. The fields required for Dirichlet zeta functions are the field of p-adic numbers and its unramified quadratic extension for every prime p as well as the field of real numbers and its unique quadratic extension, which is the field of complex numbers. Fourier analysis on related adelic rings permits an application of the Poisson summation formula to a proof of the functional identity.

A Dirichlet zeta function is a generalization of the gamma function, which satisfies no functional identity but which does satisfy a recurrence relation. The concept of a functional identity is subordinated to the concept of a recurrence relation in the proof of the Riemann hypothesis. The recurrence relation for the gamma function is reformulated as a positivity condition which applies to zeta functions.

Motivation for the proof of the Riemann hypothesis was supplied by David Trutt, who discovered nonnegative measures on the Borel subsets of the complex plane with respect to which the Newton polynomials

$$(-1)^n\frac{z(z-1)\dots(z+1-n)}{1\dots n}$$

are orthogonal. If $\nu > -1$, a unique Hilbert space exists whose elements are functions analytic in the half-plane $z + z^- > -1 - \nu$ and which contains the Newton polynomials as an orthogonal set with

$$\frac{(\nu+1)\dots(\nu+n)}{1\dots n}$$

as the square of the norm of the n-th polynomial. The identity

$$2\pi\Gamma(1+\nu)\|F\|^2$$
$$= \sum_{n=0}^{\infty}\Gamma(1+n)^{-1}\int_{-\infty}^{+\infty}|\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}+\tfrac{1}{2}\nu-it)\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}-\tfrac{1}{2}\nu-it)F(\tfrac{1}{2}n+\tfrac{1}{2}+\tfrac{1}{2}\nu-it)|^2dt$$

holds for every element $F(z)$ of the space. A structure theory for such spaces is obtained by David Trutt and the author, *Orthogonal Newton polynomials*, Advances in Mathematics 37 (1980), 251-271.

Related Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) exist for every nonnegative integer n. The elements of the n-th space are polynomials considered with the scalar product corresponding to the norm

$$\|F\|^2 = \int_{-\infty}^{+\infty} |\Gamma(\tfrac{1}{2}n + \tfrac{1}{2} + \tfrac{1}{2}\nu - it)\Gamma(\tfrac{1}{2}n - \tfrac{1}{2} - \tfrac{1}{2}\nu - it)F(t)|^2 dt.$$

These scalar products on polynomials have characteristic properties which are expressed in a pair of adjoint transformations: The transformation of the space of all polynomials considered with the $(n+1)$-st scalar product into the space of all polynomials considered with the n-th scalar product takes $F(z)$ into

$$(\tfrac{1}{2}n + \tfrac{1}{2} + \tfrac{1}{2}\nu - iz)F(z + \tfrac{1}{2}i).$$

The transformation of the space of all polynomials considered with the n-th scalar product into the space of all polynomials considered with the $(n+1)$-st scalar product takes $F(z)$ into

$$(\tfrac{1}{2}n - 1 - \tfrac{1}{2}\nu - iz)F(z + \tfrac{1}{2}i).$$

An axiomatic treatment of the spaces is given by David Trutt and the author, *Meixner and Pollaczek spaces of entire functions*, Journal of Mathematical Analysis and Applications 22 (1968), 12–24.

The weight functions which appear are reciprocals of weight functions appearing in the theory of Sonine spaces. The properties of Pollaczek polynomials are suggestive of a general theory which includes the Sonine spaces and the spaces of entire functions appearing in the Maass theory. The half-unit spacing which appears in measures is significant for the Riemann hypothesis as the spacing between the critical line and the boundary of the critical strip. The spaces of the Maass theory are unsatisfactory for the Riemann hypothesis because the spectral line is not the critical line but the right boundary of the critical strip. Related Hilbert spaces of entire functions are wanted in which the spectral line is shifted one-half unit to the left. A mechanism is suggested for making such a shift.

Related motivation for the proof of the Riemann hypothesis was supplied in 1961 by Arne Beurling and Paul Malliavin at an International Symposium on Functional Analysis held at Stanford University. Their results *On Fourier transforms of measures with compact support* appear in Acta Mathematica 107 (1962), 291-392. A source of their work is the Colloquium Publication of Norman Levinson on *Gap and Density Theorems,* American Mathematical Society, 1940. Beurling and Malliavin solve a problem of Levinson which can be formulated in the theory of Hilbert spaces of entire functions.

The problem concerns properties of a maximal totally ordered family of Hilbert spaces of entire functions. Assume that the defining functions $E(t, z)$ are parameterized by the positive numbers t in such a way that the space with index a is less than or equal to the space with index b when a is less than or equal to b. The ratio

$$E(b, z)/E(a, z)$$

is then analytic and of bounded type in the upper half-plane. A nondecreasing function $\tau(t)$ of positive numbers t exists such that the mean type of the ratio is equal to

$$\tau(b) - \tau(a)$$

when a is less than or equal to b. A computation of mean type is made from a phase function $\phi(a,x)$ for $E(a,z)$ and a phase function $\phi(b,x)$ for $E(b,z)$ as the limit of

$$[\phi(b,x) - \phi(a,x)]/x$$

as x converges to infinity on the positive or the negative half-line. The defining function $E(t,z)$ can be chosen with phase function $\phi(t,x)$, which vanishes at the origin, so that

$$\phi(t,x)/x$$

is a nondecreasing function of positive numbers t with limit zero as t converges to zero for every real number x. The inequality

$$\tau(b) - \tau(a) \leq \liminf \phi(b,x)/x$$

holds as x converges to infinity on the positive or the negative half-line when a is less than or equal to b.

A classical problem of spectral theory is formulated as a determination of the relationship between the asymptotic behavior of phase functions and the mean type of ratios of defining functions. Assume that a real number τ less than $\tau(b)$ is given such that the inequality

$$\tau(b) - \tau < \liminf \phi(b,x)/x$$

holds as x converges to infinity on the right and left half-lines. The problem is to determine whether a member $\mathcal{H}(E(a))$ of the family exists such that

$$\tau(a) = \tau.$$

An affirmative answer is given for a positive number τ less than $\tau(b)$ when $\phi(b,x)$ is a uniformly continuous function of x such that the integral

$$\int_{-\infty}^{+\infty} \frac{|\phi(b,x) - \tau(b)x| dx}{1 + x^2}$$

is finite.

The problem is a special case of an inverse spectral problem due to Mark Krein. Consider special Hilbert spaces of entire functions which are symmetric about the origin: An isometric transformation of the space into itself is defined by taking $F(z)$ into $F(-z)$. The condition is obviously satisfied when the defining function $E(z)$ of a space $\mathcal{H}(E)$ satisfies the symmetry condition

$$E^*(z) = E(-z).$$

A converse result is true. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3), which is symmetric about the origin, and which contains a nonzero element, is isometrically equal to a space $\mathcal{H}(E)$ for an entire function $E(z)$ which satisfies the symmetry condition. The spectral theory of the vibrating string is contained in structure theory of maximal totally ordered families of Hilbert spaces of entire functions which are symmetric about the origin. The function $\tau(t)$ for such a family determines the length of the string. The inverse problem of Mark Krein is the determination not only of the length but also of the mass distribution of the string.

Work on the Riemann hypothesis was interrupted during more than four years of effort required to complete the proof of the Bieberbach conjecture. The attack on the Riemann hypothesis was resumed after the confirmation of the proof in 1984. An invitation to address the Winter Meeting of the American Mathematical Society was used to present *The Riemann hypothesis for Hilbert spaces of entire functions*, which was published in the Bulletin of the American Mathematical Society 15 (1986), 1-17. An axiomatization is made of the theory of the gamma function. The positivity condition which is introduced implies the Riemann hypothesis if it applies to Dirichlet zeta functions.

The concept of a quantum gamma function with quantum q applies when q is a given number, $0 < q < 1$. A weight function is a function which is analytic and without zeros in the upper half-plane. The weighted Hardy space associated with such a function $W(z)$ is the set of functions $F(z)$, analytic in the upper half-plane, such that $F(z)/W(z)$ is of bounded type and of nonpositive mean type in the half-plane and has square integrable boundary values on the real axis. A Hilbert space $\mathcal{F}(W)$ is obtained in the norm

$$\|F\|^2_{\mathcal{F}(W)} = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt.$$

The given weight function is said to be a quantum gamma function with quantum q if the space is well-related to the transformation which takes $F(z)$ into $F(z + i\kappa)$ for every positive number κ such that

$$q \leq \exp(-2\pi\kappa).$$

Every element of the space is of the form $F(z)+F(z+i\kappa)$ for an element $F(z)$ of the space such that $F(z + i\kappa)$ belongs to the space. The scalar product

$$\langle F(t), F(t + i\kappa)\rangle_{\mathcal{F}(W)}$$

has nonnegative real part for every such element $F(z)$. An equivalent condition is that the weight function has an analytic extension to the half-plane $-\kappa < iz^- - iz$ such that the ratio

$$W(z)/W(z + i\kappa)$$

has nonnegative real part in the half-plane.

The concept of a quantum gamma function supplies an alternative to the Beurling-Malliavin theorem. A maximal totally ordered family of Hilbert spaces of entire functions exists with these properties: A function $E(t, z)$ which defines an element of the family is of Pólya class and the ratio

$$E(t, z)/W(z)$$

is of bounded type in the upper half-plane. If $\tau(t)$ is the mean type of the ratio in the half-plane, then multiplication by

$$\exp[i\tau(t)z]$$

is a contractive transformation of the space $\mathcal{H}(E(t))$ into the space $\mathcal{F}(W)$ which is isometric on the domain of multiplication by z. If κ is a positive number such that

$$q \leq \exp(-2\pi\kappa),$$

then every element of the space $\mathcal{H}(E(a))$ is of the form $F(z) + F(z + i\kappa)$ for an element $F(z)$ of the space such that $F(z + i\kappa)$ belongs to the space, and the scalar product

$$\langle F(t), F(t + i\kappa)\rangle_{\mathcal{H}(E(a))}$$

has nonnegative real part which originates with Carleman. For every positive number τ such that

$$W(z)\exp(i\tau z)$$

is unbounded on the upper half of the imaginary axis, a member of the family with defining function $E(a,z)$ exists such that $\tau(a) = \tau$.

The concept of a quantum gamma function validates a formal theory of quasi-analyticity which originates with Carleman. The issue is treated in the context of Fourier analysis by Norman Levinson in his Colloquium Publication on *Gap and Density Theorems,* American Mathematical Society, 1940. A related treatment of quasi-analyticity is given by the author in his thesis, *Local operators on Fourier transforms*, Duke Mathematical Journal 25 (1958), 143-153. If $K(x)$ is a measurable function of real x, define an operator on absolutely convergent Fourier transforms which takes

$$f(x) = \int_{-\infty}^{+\infty} F(t)\exp(2\pi i\, xt)dt$$

into

$$g(x) = \int_{-\infty}^{+\infty} G(t)\exp(2\pi i\, xt)dt$$

whenever the identity

$$G(t) = K(t)F(t)$$

holds for almost all real t with finiteness of the integrals

$$\int_{-\infty}^{+\infty} |F(t)|dt$$

and

$$\int_{-\infty}^{+\infty} |G(t)|dt.$$

If the integral

$$\int_{-\infty}^{+\infty} \frac{\log(1+|K(t)|^2)}{1+t^2}dt$$

is infinite and if a smoothness hypothesis is satisfied, then the domain of the operator contains no function which vanishes in an interval without vanishing identically. A sufficient smoothness condition is that the logarithm of

$$1+|K(t)|^2$$

is a uniformly continuous function of t. The search for an optimal smoothness hypothesis is a fundamental problem of the Carleman theory which motivates the concept of a quantum gamma function.

The quantum generalization of the gamma function is an attack on the Riemann hypothesis since the desired location of zeros is a consequence of the positivity condition characteristic of quantum gamma functions. Examples of weight functions which satisfy the positivity conditions appear in the Sonine theory. The weight function for the Hankel transformation of order ν is

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

A quantum gamma function is obtained when ν is nonnegative. A proof of positivity is given from properties of the Laplace transformation. If ν is a nonnegative number, define $\mathcal{D}_\nu$ to be the Hilbert space of functions $F(z)$, analytic in the upper half-plane, which are of the form

$$F(z) = \int_0^\infty f(t) t^{1+\nu} \exp(\pi i t^2 z) dt$$

for a measurable function $f(t)$ of positive numbers t such that the integral

$$\|F\|_{\mathcal{D}_\nu}^2 = \int_0^\infty |f(t)|^2 t \, dt$$

is finite. When ν is zero, the identity

$$\|F\|_{\mathcal{D}_\nu}^2 = \sup \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx$$

is satisfied with the least upper bound taken over all positive numbers y. When ν is positive, the identity

$$\Gamma(\nu) \|F\|_{\mathcal{D}_\nu}^2 = (2\pi)^\nu \int_0^\infty \int_{-\infty}^{+\infty} |F(x+iy)|^2 y^{\nu-1} dx \, dy$$

is satisfied. An element $F(z)$ of the space $\mathcal{D}_\nu$ is the Laplace transform of a function $f(t)$ which vanishes in an interval $(0, a)$ containing the origin if, and only if,

$$\exp(-\pi i a^2 z) F(z)$$

converges to zero as z converges to infinity on the upper half of the imaginary axis.

The Mellin transform of an element $f(z)$ of the space $\mathcal{D}_\nu$ is the function $F(x)$ of real x which is defined by

$$F(x) = \int_0^\infty f(it) t^{\frac{1}{2}\nu - ix - \frac{1}{2}} dt$$

when the integral is absolutely convergent. The identity

$$\pi^\nu \int_{-\infty}^{+\infty} |F(x)/W(x)|^2 dx = \|f(z)\|_{\mathcal{D}_\nu}^2$$

is then satisfied with

$$W(z) = \pi^{-\frac{1}{2}\nu - \frac{1}{2} + iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

The transformation is extended to the space $\mathcal{D}_\nu$ so as to maintain the identity. Every measurable function $F(x)$ of real x for which the integral converges is the Mellin transform of an element of the space $\mathcal{D}_\nu$. If the function

$$\exp(-\pi i z) f(z)$$

converges to zero as z converges to infinity on the upper half of the imaginary axis, an element $F(z)$ of the space $\mathcal{F}(W)$ is defined by

$$F(z) = \int_0^\infty f(it) t^{\frac{1}{2}\nu - iz - \frac{1}{2}} dt$$

when z is in the upper half-plane. The identity

$$\pi^\nu \|F\|^2_{\mathcal{F}(W)} = \|f(z)\|^2_{\mathcal{D}_\nu}$$

is satisfied. Every element of the space $\mathcal{D}_\nu$ is of this form. The function

$$F(z + i\kappa) = \int_0^\infty t^\kappa f(it) t^{\frac{1}{2}\nu - iz - \frac{1}{2}} dt$$

belongs to the space $\mathcal{F}(W)$ if, and only if,

$$(-iz)^\kappa f(z)$$

also belongs to the space $\mathcal{D}_\nu$. The scalar product

$$\langle F(t), F(t + i\kappa)\rangle_{\mathcal{F}(W)}$$

has nonnegative real part when $0 < \kappa < 1$ because it is a positive multiple of the scalar product

$$\langle f(z), (-iz)^\kappa f(z)\rangle_{\mathcal{D}_\nu}$$

which has nonnegative real part.

A verification that $W(z)$ is a quantum gamma function with quantum

$$q = \exp(-2\pi)$$

is thereby obtained from a spectral theory of the shift operator. The operator is unitarily equivalent to a multiplication operator in a space of functions analytic in the upper half-plane with norm defined by integration with respect to a nonnegative plane measure. The desired positivity properties of the shift operator result from the positivity properties of the multiplication operator.

The proof of the Riemann hypothesis verifies a positivity condition only for those Dirichlet zeta functions which are associated with nonprincipal real characters. The classical zeta function does not satisfy a positivity condition since the condition is not compatible with the singularity of the function. But a weaker condition is satisfied which has the desired implication for zeros.

A curious coincidence needs to be mentioned as part of the chain of events which concluded in the proof of the Riemann hypothesis. The feudal family de Branges originates in a crusader who died in 1199 leaving an emblem of three swords hanging over three coins, surmounted by the traditional crown designating a count, and inscribed with the motto "Nec vi nec numero." This is a citation from Chapter 4, Verse 6, of the Book of Zechariah: "Not by might, nor by power, but by my Spirit, says the Lord of Hosts." The château de Branges was destroyed in 1478 by the army of Louis XI of France during an unsuccessful campaign to wrest Franche-Comté from the heirs of Charles the Bold of Burgundy. The family de Branges performed administrative, legal, and religious functions in Saint-Amour for the marquisat d'Andelôt during Spanish rule of Franche-Comté. François de Branges of Saint-Amour received the seigneurie de Bourcia in 1679 when Franche-Comté became part of France. The château de Bourcia remained the home of his descendants until it was destroyed by Parisian revolutionaries in 1791. The château d'Andelôt near Saint-Amour, which survived the revolution, was bought in 1926 by Pierre du Pont, an elder brother of Irénée du Pont, for a nephew assigned in diplomatic service to France. This coincidence accounts for the interest which Irénée du Pont showed in a student of mathematics. The funds for his undergraduate education are conjectured to have been secretly donated by Mr. du Pont. The restoration of the château de Bourcia as a site dedicated to analysis, not only in mathematics, is suggested by the remarkable events which culminate in the proof of the Riemann hypothesis.